APPM 2350 **EXAM 2** **A Spring when you were in grade school**

INSTRUCTIONS: Electronic devices, books, and crib sheets are not permitted. Write your (1) name, (2) instructor's name, and (3) recitation number on the front of your bluebook. Work all problems. Show your work clearly. Note that a correct answer with incorrect or no supporting work may receive no credit, while an incorrect answer with relevant work may receive partial credit.

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1. (25 Points) Near Kate's Lake in a certain well–known National Park, the elevation of the solid ground (in feet) can be described by the function $f(x, y) = 8000 - 30x^2y^2 + 30x^2 + 30y^2$.

   (a) Determine the x, y, and z coordinates corresponding to the location of the bottom of Kate's Lake.

   (b) Determine the x, y, and z coordinates of any possible location(s) from which Kate's Lake might drain.

   (c) What is the maximum possible depth of Kate's Lake? If is not possible to determine this from the given information, clearly state "Cannot be determined".

2. (20 points) FROSTY is a probe to measure variations in ice trapped in the surface of Martian soil. It orbits above the Martian poles and uses lasers aimed straight down at the surface to measure variations in the water content per unit area (million molecules per square centimeter, which is abbreviated MMPSC). The search pattern on the ground is approximated by a nice xy-grid. You may assume that a local xy-coordinate system is placed with its origin on a point P. As FROSTY's laser passes over the point P in the $\mathbf{i}$ direction the rate of change in water content with respect to distance traveled was measured at 5 MMPSC/meter. On a later pass over point P while moving in the $\mathbf{j}$ direction the rate of change in water content with respect to distance traveled was measured at 12 MMPSC/meter.

   (a) Assume that at point P the actual water concentration is 50 MMPSC. If one considers a circular region of radius one meter centered on point P, estimate the minimum and maximum values of water concentration (in MMPSC) that would be found in the circular patch.

   (b) Determine the coordinates of the minimum and maximum MMPSC values in the circular patch.

3. (20 points) Consider the curve C in the xy-plane defined by $y = (x-1)^2$.

   (a) Using an appropriate Calculus III technique, determine the coordinates of any points on C such that $x + y^2$ is as small as possible.

   (b) Draw the constraint curve and several of the level curves of the objective function (the function you are trying to maximize or minimize) in the xy-plane.

   (c) Based on your results from part (a), determine the minimum value(s) of $x + y^2$ on curve C.

   (d) On the graph label all points of interest on the constraint curve. At each of these points, clearly indicate the value of the objective function and whether it is a local maximum or minimum.

4. (20 points) Consider the function $f(x, y) = x^2 + y^2 + x^4y^4$.

   (a) Calculate the *second order* Taylor approximation to $f(x, y)$ near the point $(1, 1)$.

   (b) Use your result from part (a) to estimate the value of $f(1.1, 1.2)$.

   (c) Calculate an "upper bound on the error" associated with your *second order* approximation assuming that you only use values of x and y such that $|x - 1| \leq 0.1$ and $|y - 1| \leq 0.2$.

5. (25 Points) Suppose you are to evaluate f where $f = xy^2$. However it turns out that $x/y = z$, so you can actually calculate f three different ways:

$$\text{(a)} \quad f = xy^2 \qquad\qquad \text{(b)} \quad f = zy^3 \qquad\qquad \text{(c)} \quad f = x^3/z^2\,.$$

Unfortunately you put in the wrong values for each variable. Specifically, your value for x is 1% high, your y is 2% high, and your z is 3% high.

(a) Determine the resulting percent error in f for each of the three calculation methods. Be sure to clearly indicate your result for methods (a), (b), and (c). Also, be sure to indicate if each result is high or low.

(b) If you could reduce the percent error in one of the variables x, y, or z by 1/2, which one would you reduce so that you could reduce the final error in your calculation of f by as much as possible? Be sure to state what the new final percent error in f would be, and which calculation method (a, b, or c) should now be used.

Projections and distances

$$\mathrm{proj}_{\mathbf{A}}\mathbf{B} = \left(\frac{\mathbf{A}\cdot\mathbf{B}}{\mathbf{A}\cdot\mathbf{A}}\right)\mathbf{A} \qquad d = \frac{|\overrightarrow{PS}\times\mathbf{v}|}{|\mathbf{v}|} \qquad d = \left|\overrightarrow{PS}\cdot\frac{\mathbf{n}}{|\mathbf{n}|}\right|$$

Arc length, frenet formulas, and tangential and normal acceleration components

$$ds = |\mathbf{v}|\,dt \qquad \mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{\mathbf{v}}{|\mathbf{v}|} \qquad \mathbf{N} = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} \qquad \mathbf{B} = \mathbf{T}\times\mathbf{N}$$

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N} \qquad \frac{d\mathbf{B}}{ds} = -\tau\mathbf{N} \qquad \kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \frac{|f''(x)|}{|\,1+(f'(x))^2\,|^{3/2}} = \frac{|\dot{x}\ddot{y}-\dot{y}\ddot{x}|}{|\dot{x}^2+\dot{y}^2|^{3/2}} \qquad \tau = -\frac{d\mathbf{B}}{ds}\cdot\mathbf{N}$$

$$\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T} \qquad a_T = \frac{d|\mathbf{v}|}{dt} \qquad a_N = \kappa|\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}$$

Directional derivative, discriminant, and Lagrange multipliers

$$\frac{df}{ds} = (\nabla f)\cdot\mathbf{u} \qquad f_{xx}f_{yy} - (f_{xy})^2 \qquad \nabla f = \lambda\nabla g, \quad g = 0$$

Taylor's formula (at the point (x_0, y_0))

$$\begin{aligned}
f(x,y) \quad = \quad & f(x_0,y_0) + \Big[\,(x-x_0)f_x(x_0,y_0) + (y-y_0)f_y(x_0,y_0)\Big] \\
& + \frac{1}{2!}\Big[\,(x-x_0)^2 f_{xx}(x_0,y_0) + 2(x-x_0)(y-y_0)f_{xy}(x_0,y_0) + (y-y_0)^2 f_{yy}(x_0,y_0)\Big] \\
& + \frac{1}{3!}\Big[\,(x-x_0)^3 f_{xxx}(x_0,y_0) + 3(x-x_0)^2(y-y_0)f_{xxy}(x_0,y_0) \\
& \qquad + 3(x-x_0)(y-y_0)^2 f_{xyy}(x_0,y_0) + (y-y_0)^3 f_{yyy}(x_0,y_0)\Big] + \cdots
\end{aligned}$$

Linear approximation error

$$|E(x,y)| \le \frac{1}{2}M(|x-x_0| + |y-y_0|)^2, \qquad \text{where } \max\{\,|f_{xx}|,\,|f_{xy}|,\,|f_{yy}|\,\} \le M$$