INSTRUCTIONS: Electronic devices, books, and crib sheets are not permitted. Write your (1) name, (2) instructor's name, and (3) recitation number on the front of your bluebook. Work all problems. Show your work clearly. Note that a correct answer with incorrect or no supporting work may receive no credit, while an incorrect answer with relevant work may receive partial credit.

1. (25 points) Consider a surface with height h described by the function $h(x, y) = x\, e^{-(x^2+y^2)}$.

   (a) Determine the location of all critical points of $h(x, y)$.

   (b) Classify all critical points.

   (c) What are the local extreme values of h?

   (d) If you were to stand on the surface at the point P corresponding to $x = 0$ and $y = 0$, in which direction(s) would you walk to increase your elevation most rapidly? What would the rate of increase be?

   (e) Again at point P, in which direction(s) would you travel to keep your elevation constant?

2. (25 Points) You are doing some consulting for a car company that manufactures three colors of cars, black, blue, and red. Let x, y, and z denote the number of black, blue, and red cars manufactured. After doing some research, you have determined that the total sales the company brings in each year in dollars is approximated well by the function xyz^2, and it costs the company \$10,000 to make each car, no matter what the color is. The factory in which the cars are produced can manufacture 8000 cars per year, and all vehicles that are manufactured are sold.

   (a) Determine the equation for the profit that the company makes in a year. Recall that the profit is given by the total sales minus the expenses.

   (b) Write down the equation that constrains the amount of profit that the company can make in a year.

   (c) Determine the number of black, blue, and red cars that the company should manufacture to maximize their profits.

3. (25 Points) Farnsworth Fruit Fly III is from a wealthy fruit fly family that made its fortune from their business F4FF (Fones 4 Fruit Flies). While flying along, Farnsworth is usually too busy playing on his phone to notice what is ahead. Oops, he just flew over a candle flame located at $(0, 0)$ and burned off part of his left wing. He can no longer fly straight. Too bad for Farnsworth. Instead, he now flies along the path $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j}$ for $0 \le t \le 2$. The temperature in the room is described by the function $T(x, y) = xy$.

   (a) When, and where, does Farnsworth see the maximum value of dT/dt? What is the maximum value of dT/dt?

   (b) Let s be the distance traveled by Farnsworth as he flies along. When, and where, does Farnsworth see the maximum value of dT/ds? What is the maximum value of dT/ds?

   (c) The next day Farnsworth returns to the scene of his accident but he now flies half as fast along the same path for $0 \le t \le 2$. Will either of the maximum values of dT/dt or dT/ds change? If a value changes, clearly state by how much it changes *and support your answer.*

No live fruit flies were harmed in the development of this problem or its solution.

Farnsworth's family will eventually successfully sue their own company for damages and Farnsworth will get a prosthetic wing tip.

4. (25 points) Consider the function $f(x, y) = x^3\,y^3 + x^2\,y^2$.

(a) Calculate the *second order* Taylor approximation to $f(x, y)$ near the point $(1, 1)$.

(b) Use your result from part (a) to estimate the value of $f(1.1, 1.1)$.

(c) Calculate an "upper bound on the error" associated with this *second order* approximation assuming that you only use values of x and y such that $|x - 1| \leq 0.1$ and $|y - 1| \leq 0.1$.

Projections and distances

$$\mathrm{proj}_{\mathbf{A}}\mathbf{B} = \left(\frac{\mathbf{A}\cdot\mathbf{B}}{\mathbf{A}\cdot\mathbf{A}}\right)\mathbf{A} \qquad d = \frac{|\overrightarrow{PS}\times\mathbf{v}|}{|\mathbf{v}|} \qquad d = \left|\overrightarrow{PS}\cdot\frac{\mathbf{n}}{|\mathbf{n}|}\right|$$

Arc length, frenet formulas, and tangential and normal acceleration components

$$ds = |\mathbf{v}|\,dt \qquad \mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{\mathbf{v}}{|\mathbf{v}|} \qquad \mathbf{N} = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} \qquad \mathbf{B} = \mathbf{T}\times\mathbf{N}$$

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N} \qquad \frac{d\mathbf{B}}{ds} = -\tau\mathbf{N} \qquad \kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \frac{|f''(x)|}{|\,1+(f'(x))^2\,|^{3/2}} = \frac{|\dot{x}\ddot{y}-\dot{y}\ddot{x}|}{|\dot{x}^2+\dot{y}^2|^{3/2}} \qquad \tau = -\frac{d\mathbf{B}}{ds}\cdot\mathbf{N}$$

$$\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T} \qquad a_T = \frac{d|\mathbf{v}|}{dt} \qquad a_N = \kappa|\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}$$

Directional derivative, discriminant, and Lagrange multipliers

$$\frac{df}{ds} = (\nabla f)\cdot\mathbf{u} \qquad f_{xx}f_{yy} - (f_{xy})^2 \qquad \nabla f = \lambda\nabla g, \quad g = 0$$

Taylor's formula (at the point (x_0, y_0))

$$\begin{aligned} f(x,y) \quad = \quad & f(x_0,y_0) + \Big[\,(x-x_0)f_x(x_0,y_0) + (y-y_0)f_y(x_0,y_0)\,\Big] \\ & + \frac{1}{2!}\Big[\,(x-x_0)^2 f_{xx}(x_0,y_0) + 2(x-x_0)(y-y_0)f_{xy}(x_0,y_0) + (y-y_0)^2 f_{yy}(x_0,y_0)\,\Big] \\ & + \frac{1}{3!}\Big[\,(x-x_0)^3 f_{xxx}(x_0,y_0) + 3(x-x_0)^2(y-y_0)f_{xxy}(x_0,y_0) \\ & \qquad + 3(x-x_0)(y-y_0)^2 f_{xyy}(x_0,y_0) + (y-y_0)^3 f_{yyy}(x_0,y_0)\,\Big] + \cdots \end{aligned}$$

Linear approximation error

$$|E(x,y)| \le \frac{1}{2}M(|x-x_0| + |y-y_0|)^2, \qquad \text{where } \max\{\,|f_{xx}|,\,|f_{xy}|,\,|f_{yy}|\,\} \le M$$