INSTRUCTIONS: Electronic devices, notes, books, and crib sheets are not permitted. Write your (1) name, (2) instructor's name, and (3) lecture number on the front of your bluebook. Work all problems. Show your work clearly. Note that a correct answer with incorrect, or insufficient supporting work may receive no credit, while an incorrect answer with relevant work may receive partial credit. Help your graders help you!

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1. (25 points) Around Itsa Lake the elevation, h, of the solid ground (measured in feet) can be described by the function $h(x, y) = 9000 - 20x^2y^2 + 40x^2 + 40y^2$.

   (a) Determine the x, y, and h coordinates corresponding to the location of the bottom of Itsa Lake.
   SOLUTION: We first calculate $h_x = x(-40y^2 + 80)$ and $h_y = y(-40x^2 + 80)$. There are no locations for which h_x and/or h_y are undefined. We then look for critical points (CPs) associated with $h_x = 0$ and $h_y = 0$ (simultaneously). This leads to the following five CPs, $P_0(0, 0)$, $P_1(\sqrt{2}, \sqrt{2})$, $P_2(-\sqrt{2}, \sqrt{2})$, $P_3(-\sqrt{2}, -\sqrt{2})$, and $P_4(\sqrt{2}, -\sqrt{2})$.
   We now need to classify the CPs. First, $h_{xx} = -40y^2+80$, $h_{yy} = -40x^2+80$, and $h_{xy} = -80xy$. Hence the discriminant is

   $$D = h_{xx}h_{yy} - h_{xy}^2 = (-40y^2 + 80)(-40x^2 + 80) - (-80xy)^2 .$$

   We can now evaluate D at each of the CPs to get $D|_{P_0} = 80^2 > 0$, however D evaluated at points P_1 through P_4 have the common value $D = -(160)^2 < 0$. We conclude that P_1 through P_4 are all "saddle points." At P_0 we need to further examine the sign on either f_{xx} or f_{yy} (either one will do). Since $f_{xx}|_{P_0} = 80 > 0$, we conclude that P_0 corresponds to a minimum.
   Now, if we evaluate h at all of our CPs we get $h|_{P_0} = 9000$ while h at all of the saddle points have the common value $h = 9080$.

   (b) Determine the x, y, and h coordinates of any possible location(s) from which Itsa Lake might drain.
   SOLUTION: Possible drainage locations would be our saddle points, P_1 through P_4. Evaluating h at each saddle point, we find $f = 9080$.

   (c) What is the maximum possible depth of Itsa Lake? If is not possible to determine this from the given information, clearly state "Cannot be determined."
   SOLUTION: The "bottom" of the lake has elevation $h = 9000$ while of the drainage locations have the common elevation $h = 9080$. Hence, the maximum lake depth is 80 feet.

2. (25 Points) Consider a circle of radius R centered on the origin. You need to determine the coordinates of the points on the circle closest and farthest to a point outside the circle located at $P_0(\alpha, \beta)$. You may assume that P_0 is in the first quadrant. Clearly, one could construct a line from P_0 to the center, and then move a distance R along the line in either direction from the center. But, of course, this is *not* how we want you to solve the problem. As a Calculus III student, you need to impress your graders by doing this calculation using Calculus III concepts. The higher the concept level, the higher your possible grade. Of course you will also justify and explain your reasoning as you progress through this problem. Right?

SOLUTION: The objective function is $f(x, y) = (x - \alpha)^2 + (y - \beta)^2$, which represents the distance squared. The constraint is the $g(s, h) = x^2 + y^2 = R^2$, which means the solution location (x, y) must be on the circle. Using $\nabla f = \lambda(\nabla g)$ we get

$$2(x - \alpha)\,\mathbf{i} + 2(y - \beta)\,\mathbf{j} = \lambda(2x\,\mathbf{i} + 4y\,\mathbf{j})$$

Setting the $\mathbf{i}$ and $\mathbf{j}$ components equal, and accounting for the constraint, we get the three coupled algebraic equations

$$\begin{aligned} x - \alpha &= \lambda x\,, && (1)\\ y - \beta &= \lambda y\,, && (2)\\ x^2 + y^2 &= R^2\,. && (3) \end{aligned}$$

Note that $x = 0$ cannot be a solution for x since it will not satisfy equation (1). Similar reasoning indicates that $y = 0$ cannot be a solution. Now, equating λ from (1) and (2) shows that

$$\lambda = \frac{x - \alpha}{x} = \frac{y - \beta}{y}\,,$$

from which it follows that $y = \beta x/\alpha$. Using $y = \beta x/\alpha$ in (3) easily leads to $x = \dfrac{\pm\alpha R}{\sqrt{\alpha^2 + \beta^2}}$. Now, for each of these x values, we can use $y = \beta x/\alpha$ to calculate the corresponding y value. Finally we get the two points $P_1\left(\dfrac{\alpha R}{\sqrt{\alpha^2 + \beta^2}}, \dfrac{\beta R}{\sqrt{\alpha^2 + \beta^2}}\right)$ and $P_2\left(\dfrac{-\alpha R}{\sqrt{\alpha^2 + \beta^2}}, \dfrac{-\beta R}{\sqrt{\alpha^2 + \beta^2}}\right)$. Finally, one needs to give some sort of justification for which point is closet and farthest to point P_0. If you had lots of extra time, and you really, really like algebra, you could evaluate the objective function f at P_1 and P_2 to find that P_1 is closest to P_0. You could have drawn a clear sketch. Or, you could have based your argument on the fact that P_0 and P_1 are both in the first quadrant, while P_2 is in the third quadrant. How ever you justified it, P_1 is closest to P_0.

3. (25 Points) Consider the function $F(u, v)$, where u and v are functions of x and y specifically, $u(x, y)$ and $v(x, y)$ respectively. For a particular set of F, x, y, u, and v values, $F_u = 1$, $F_v = \beta$, $u_x = \alpha$, $u_y = 2$, $v_x = 2$, $v_y = 3$, where α and β are real constants. Reread all of this to make sure you've got it all straight. Maybe one or two more times, just to make sure.

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\begin{tikzcd}
& F(u,v) \arrow[dl] \arrow[dr] & \\
u(x,y) \arrow[d] \arrow[drr] & & v(x,y) \arrow[d] \arrow[dll] \\
x & & y
\end{tikzcd}
$$

(a) Suppose you are now told that for the above conditions, $dF = 7\,dx + 8\,dy$. If x changes by the small amount 0.01 and y changes by the small amount -0.02, estimate the change in the value of F.
SOLUTION: $\Delta F \approx 7\,\Delta x + 8\,\Delta y = 7(0.01) + 8(-0.02) = -0.09$

(b) If $dF = 7\,dx + 8\,dy$ still holds, then determine the values of α and β.
SOLUTION: Since we know that $F_x = 7 = F_u u_x + F_v v_x = 1\alpha + 2\beta$ and $F_y = 8 = F_u u_y + F_v v_y = 1(2) + \beta 3$ we can solve for $\alpha = 3$ and $\beta = 2$.

(c) When x changes by the small amount 0.01, and y changes by the small amount -0.02, estimate the change in the value of u.
SOLUTION: $\Delta u \approx u_x \Delta x + u_y \Delta y = 3(0.01) + 2(-0.02) = -0.01$.

(d) When x changes by the small amount 0.01, and y changes by the small amount -0.02, estimate the change in the value of v.
SOLUTION: $\Delta v \approx v_x \Delta x + v_y \Delta y = 2(0.01) + 3(-0.02) = -0.04$.

(e) Ultimately thinking of F as a function of x and y, and if possible, determine $\nabla F = F_x\,\mathbf{i} + F_y\,\mathbf{j}$ for the conditions described in the previous parts of the problem. Otherwise clearly state "Cannot be determined."
SOLUTION: From parts a) and b), we see that $F_x = 7$ and $F_y = 8$, so then $\nabla F = 7\,\mathbf{i} + 8\,\mathbf{j}$ for the stated conditions.

4. (25 Points) Consider the function $f(x, y) = \exp(x + y)$. Warning: carefully read all the numeric values in this question, then read them again.

(a) Calculate the *second order* Taylor approximation to $f(x, y)$ near the point $(2, 1)$.
SOLUTION: Using up to and including the second-order terms in the T.S. we get

$$\begin{aligned} f(x,y) &\approx f(x_0,y_0) + \Big[(x-x_0)f_x(x_0,y_0) + (y-y_0)f_y(x_0,y_0) \Big] \\ &\quad + \frac{1}{2!}\Big[(x-x_0)^2 f_{xx}(x_0,y_0) + 2(x-x_0)(y-y_0)f_{xy}(x_0,y_0) + (y-y_0)^2 f_{yy}(x_0,y_0) \Big] \end{aligned}$$

Noting that the function and all of its partial derivatives are equal to $\exp(x + y)$, we can write the approximation as

$$\begin{aligned} f(x,y) &\approx \exp(x_0+y_0)\Big(1 + \Big[(x-x_0) + (y-y_0) \Big] \\ &\quad + \frac{1}{2!}\Big[(x-x_0)^2 + 2(x-x_0)(y-y_0) + (y-y_0)^2 \Big]\Big) \\ &= \exp(x_0+y_0)\Big(1 + \Big[(x-x_0) + (y-y_0) \Big] \\ &\quad + \frac{1}{2!}\Big[(x-x_0) + (y-y_0) \Big]^2\Big) \end{aligned}$$

Now, since $x_0 = 2$ and $y_0 = 1$, we have

$$f(x,y) \approx e^3 \left(1 + [(x-2) + (y-1)] + \frac{1}{2!}\Big[(x-2) + (y-1)\Big)^2\right)$$

(b) Use your result from part (a) to estimate the value of $f(2.2, 1.1)$. Do not simplify your answer here. For example, you can leave your answer in the form $8 + 4(3.1 - 3) + 3(4.01 - 4)$, although we really do not recommend using these numbers.
SOLUTION: Setting $x = 2.2$ and $y = 1.1$ we get

$$f(2.2, 1, 1) \approx e^3 \left(1 + [0.2 + 0.1] + [0.2 + 0.1]^2\right)$$

(c) Calculate an "upper bound on the error" associated with your *second order* approximation assuming that you only use values of x and y such that $|x - 2| \leq 0.2$ and $|y - 1| \leq 0.1$. Please simplify your answer, but don't try to convert to decimal form.
SOLUTION: We need to determine $\max\{\, |f_{xxx}|,\ |f_{xxy}|,\ |f_{xyy}|,\ |f_{yyy}|\} \leq M$, in the region bounded by $|x - 2| \leq 0.2$ and $|y - 1| \leq 0.1$. But since all derivatives are $\exp(x + y)$, we can use $M = e^{2.2+1.1} = e^{3.3}$. Then an upper bound on the error would be

$$|error| <= \frac{e^{3.3}}{3!}\,[\,|2.2-2| + |1.1-1|\,]^3 = \frac{e^{3.3}}{3!}[0.3]^3$$

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(d) Now suppose you actually worked out the *fifth order* Taylor approximation to $f(x, y)$ near the point $(3, 2)$. (You don't actually need to work out this approximation! Also note the change in the center location from $(2, 1)$ to $(3, 2)$.) Calculate an "upper bound on the error" associated with this *fifth order* approximation assuming that you only use values of x and y such that $|x - 3| \leq 0.1$ and $|y - 2| \leq 0.1$. Please simplify your answer, but don't try to convert to decimal form.
SOLUTION: The line of reasoning is similar to part c), except the error is now based on the maximum magnitude of all possible *sixth* order derivatives in the region bounded by $|x - 3| \leq 0.1$ and $|y - 2| \leq 0.1$. This would now lead to $M = e^{3.1+2.1} = e^{5.2}$. Then an upper bound on the error would be

$$|error| <= \frac{e^{5.2}}{6!}\,[\,|3.1-3| + |2.1-2|\,]^6 = \frac{e^{5.2}}{6!}[0.2]^6$$

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Projections and distances

$$\text{proj}_{\mathbf{A}}\mathbf{B} = \left(\frac{\mathbf{A}\cdot\mathbf{B}}{\mathbf{A}\cdot\mathbf{A}}\right)\mathbf{A} \qquad d = \frac{|\overrightarrow{PS}\times\mathbf{v}|}{|\mathbf{v}|} \qquad d = \left|\overrightarrow{PS}\cdot\frac{\mathbf{n}}{|\mathbf{n}|}\right|$$

Arc length, frenet formulas, and tangential and normal acceleration components

$$ds = |\mathbf{v}|\,dt \qquad \mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{\mathbf{v}}{|\mathbf{v}|} \qquad \mathbf{N} = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} \qquad \mathbf{B} = \mathbf{T}\times\mathbf{N}$$

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N} \qquad \frac{d\mathbf{B}}{ds} = -\tau\mathbf{N} \qquad \kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \frac{|f''(x)|}{|\,1+(f'(x))^2\,|^{3/2}} = \frac{|\dot{x}\ddot{y}-\dot{y}\ddot{x}|}{|\dot{x}^2+\dot{y}^2|^{3/2}} \qquad \tau = -\frac{d\mathbf{B}}{ds}\cdot\mathbf{N}$$

$$\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T} \qquad a_T = \frac{d|\mathbf{v}|}{dt} \qquad a_N = \kappa|\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}$$

Directional derivative, discriminant, and Lagrange multipliers

$$\frac{df}{ds} = (\nabla f)\cdot\mathbf{u} \qquad f_{xx}f_{yy} - (f_{xy})^2 \qquad \nabla f = \lambda\nabla g, \quad g = 0$$

Taylor's formula (at the point (x_0, y_0))

$$\begin{aligned} f(x,y) \quad = \quad & f(x_0,y_0) + \Big[\,(x-x_0)f_x(x_0,y_0) + (y-y_0)f_y(x_0,y_0)\Big] \\ & + \frac{1}{2!}\Big[\,(x-x_0)^2 f_{xx}(x_0,y_0) + 2(x-x_0)(y-y_0)f_{xy}(x_0,y_0) + (y-y_0)^2 f_{yy}(x_0,y_0)\Big] \\ & + \frac{1}{3!}\Big[\,(x-x_0)^3 f_{xxx}(x_0,y_0) + 3(x-x_0)^2(y-y_0)f_{xxy}(x_0,y_0) \\ & \qquad + 3(x-x_0)(y-y_0)^2 f_{xyy}(x_0,y_0) + (y-y_0)^3 f_{yyy}(x_0,y_0)\Big] + \cdots \end{aligned}$$

Linear approximation error

$$|E(x,y)| \le \frac{M}{2}(|x-x_0| + |y-y_0|)^2, \qquad \text{where } \max\{\,|f_{xx}|,\, |f_{xy}|,\, |f_{yy}|\,\} \le M$$